High-Order Summation-By-Parts Operators on Unstructured Grids

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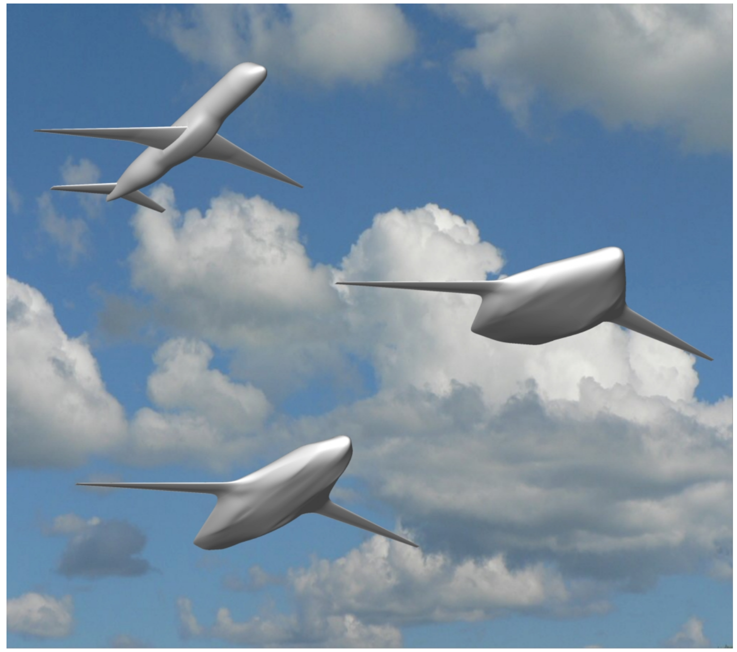
My Research Group

Research underway in the following areas:

- high-order discretization methods in time and space
  - summation-by-parts operators
  - implicit Runge-Kutta methods
- algorithms for aerodynamic shape optimization
  - two-level free-form deformation geometry control
  - integrated mesh movement method
  - enables substantial geometric flexibility (very large shape changes), analytical geometry representation
- application of aerodynamic shape optimization to unconventional energy-efficient aircraft configurations
- active flow control for drag reduction

Lifting Fuselage Configuration for Regional Class Aircraft



- shape found through RANS-based aerodynamic shape optimization
- narrower than classical hybrid wing-body
- centerbody carries about 30% of the lift (more than D8 for example)
- drag savings estimated at 6-10% relative to optimized conventional aircraft

Some recent papers that might be of interest

Reist, T.A., and Zingg, D.W., "High-Fidelity Aerodynamic Shape Optimization of a Lifting-Fuselage Concept for Regional Aircraft," J. of Aircraft, Vol. 54, No. 3, 2017, pp. 1085-1097.

Reist, T.A., and Zingg, D.W., "Multi-Fidelity Optimization of Hybrid Wing-Body Aircraft with Stability and Control Requirements," in preparation, 2017.

Rashad, R., and Zingg, D.W., "Aerodynamic Shape Optimization for Natural Laminar Flow Using a Discrete Adjoint Method," AIAA J., Vol. 54, No. 11, 2016, pp. 3321-3337.

Gagnon, H., and Zingg, D.W., "Euler-Equation-Based Drag Minimization of Unconventional Aircraft Configurations," J. of Aircraft, Vol. 53, No. 5, 2016, pp. 1361-1371.

Gagnon, H., and Zingg, D.W., "Aerodynamic Trade Study of a Box-Wing Aircraft Configuration," J. of Aircraft, Vol. 53, No. 4, 2016, pp. 971-981.

Zhang, Z.J., Khosravi, S., and Zingg, D.W., "High-fidelity aerostructural optimization with integrated geometry parameterization and mesh movement," Struct. Multidisc. Optim., Vol. 55, No. 4, 2017, pp. 1217-1235.

Boom, P.D., and Zingg, D.W., "Optimization of High-Order Diagonally Implicit Runge-Kutta Methods," in preparation, 2017.

High-Order Methods

- huge potential efficiency benefits
- popular for large eddy and direct simulation of turbulent flows
- most production Reynolds-averaged Navier-Stokes solvers are second order
- singularities and discontinuities such as shocks are an issue
- robustness is an issue

Summation by Parts (SBP) Operators

- Operators constructed specifically to satisfy the SBP property
  - leads to provable time stability
  - usually used with simultaneous approximation terms (SATs) for weak imposition of boundary and interface conditions

- Other known operators satisfy the SBP property
  - some discontinuous Galerkin (DG) operators
  - some correction procedure by reconstruction (CPR) operators
  - some spectral collocation operators

“Discontinuous” Approaches

- Includes DG, CPR, and SBP-SAT approaches
  - SBP-SAT approach can also be implemented in a continuous manner
- High order achieved through interior degrees of freedom
- Have some nice properties that can be advantageous:
  - parallelization on modern computer architectures
  - reduced need for smooth meshes
- Potential disadvantage:
  - need for high order meshes

Classical (pre-2014) Summation by Parts Operators

$$\frac{\partial \mathcal{U}}{\partial x} \approx D\boldsymbol{u} = H^{-1}Q\boldsymbol{u}$$

where H is a (diagonal) symmetric positive-definite (SPD) matrix and

$$Q + Q^T = E = \begin{bmatrix} -1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix}$$

The norm matrix H defines an inner product, norm, and quadrature as

$$(\boldsymbol{u},\boldsymbol{v})_H = \boldsymbol{u}^T H \boldsymbol{v}\ ,\ ||\boldsymbol{u}||_H^2 = \boldsymbol{u}^T H \boldsymbol{u}\ ,\ \int_{x_\mathrm{L}}^{x_\mathrm{R}} \mathcal{U}\mathrm{d}x \approx \mathbf{1}^T H \boldsymbol{u}\ ,\ \int_{x_\mathrm{L}}^{x_\mathrm{R}} \mathcal{U}\mathcal{V}\mathrm{d}x \approx \boldsymbol{u}^T H \boldsymbol{v}$$

Integration by Parts

IBP (x derivative):

$$\int_\Omega \mathcal{V}\frac{\partial \mathcal{U}}{\partial x}\mathrm{d}\Omega + \int_\Omega \mathcal{U}\frac{\partial \mathcal{V}}{\partial x}\mathrm{d}\Omega = \oint_\Gamma \mathcal{V}\mathcal{U}n_x \mathrm{d}\Gamma$$

Let $\mathcal{V} = \mathcal{U}$:

$$2\int_\Omega \mathcal{U}\frac{\partial \mathcal{U}}{\partial x}\mathrm{d}\Omega = \oint_\Gamma \mathcal{U}^2 n_x \mathrm{d}\Gamma$$

Let $\mathcal{V} = 1$:

$$\int_\Omega \frac{\partial \mathcal{U}}{\partial x}\mathrm{d}\Omega = \oint_\Gamma \mathcal{U}n_x \mathrm{d}\Gamma$$

Summing over the three components of a vector gives the divergence theorem

SBP Discretely Mimics Integration by Parts

IBP (x derivative):

$$\int_\Omega \mathcal{V}\frac{\partial \mathcal{U}}{\partial x}\mathrm{d}\Omega + \int_\Omega \mathcal{U}\frac{\partial \mathcal{V}}{\partial x}\mathrm{d}\Omega = \oint_\Gamma \mathcal{V}\mathcal{U}n_x \mathrm{d}\Gamma$$

IBP (one dimension):

$$\int_{x_L}^{x_R} \mathcal{V}\frac{\partial \mathcal{U}}{\partial x}\mathrm{d}x + \int_{x_L}^{x_R} \mathcal{U}\frac{\partial \mathcal{V}}{\partial x}\mathrm{d}x = \mathcal{V}\mathcal{U}|_{x_L}^{x_R}$$

SBP:

$$\boldsymbol{v}^T H D \boldsymbol{u} + \boldsymbol{u}^T H D \boldsymbol{v} = \boldsymbol{v}^T E \boldsymbol{u} = v_R u_R - v_L u_L$$

Setting $\boldsymbol{v} = \boldsymbol{u}$ gives

$$2\boldsymbol{u}^T H D \boldsymbol{u} = \boldsymbol{u}^T E \boldsymbol{u} = u_R^2 - u_L^2$$

Observe that this makes the role of E clear:

$$\boldsymbol{v}^T E \boldsymbol{u} \approx \oint_\Gamma \mathcal{V}\mathcal{U}n_x \mathrm{d}\Gamma$$

Linear Convection Equation: Well Posedness

Linear convection equation:

$$\frac{\partial \mathcal{U}}{\partial t} = -\frac{\partial \mathcal{U}}{\partial x}\,, \qquad x_L \le x \le x_R\,, \qquad \mathcal{U}(x_L, t) = \mathcal{U}_L(t)$$

Multiply by $2\mathcal{U}$ and integrate over the domain:

$$\int_{x_L}^{x_R} \frac{\partial \mathcal{U}^2}{\partial t}\mathrm{d}x \;\;=\;\; -\int_{x_L}^{x_R} \frac{\partial \mathcal{U}^2}{\partial x}\mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{x_R} \mathcal{U}^2\mathrm{d}x \;\;=\;\; -(\mathcal{U}_R^2 - \mathcal{U}_L^2) \;\;\le\;\; 0 \quad \text{if } \mathcal{U}_L = 0$$

Semi-Discrete Form

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -D\boldsymbol{u} = -H^{-1}Q\boldsymbol{u}$$

Multiply by $2\boldsymbol{u}^T H$:

$$\begin{aligned} 2\boldsymbol{u}^T H\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &= -2\boldsymbol{u}^T HD\boldsymbol{u} = -2\boldsymbol{u}^T HH^{-1}Q\boldsymbol{u} = -2\boldsymbol{u}^T Q\boldsymbol{u} \\ \frac{\mathrm{d}||\boldsymbol{u}||_H^2}{\mathrm{d}t} &= -2\boldsymbol{u}^T\left(\frac{Q+Q^T}{2}\right)\boldsymbol{u} = -\boldsymbol{u}^T E\boldsymbol{u} = -(u_n^2 - u_1^2) \end{aligned}$$

Mimics continuous result because

$$Q + Q^T = E = \begin{bmatrix} -1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix}$$

Classical finite-difference SBP operator ($p = 1$)

$$D = H^{-1}Q$$

Gives a first-order (p) approximation to the first derivative (at the boundary nodes) and second-order ($2p$) in the interior, which will produce a second-order ($p+1$) solution to the linear convection equation (for diagonal H)

$$H = \Delta x \begin{bmatrix} 1/2 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1/2 \end{bmatrix} \quad \text{(composite trapezoidal rule)}$$

$$Q = \begin{bmatrix} -1/2 & 1/2 & & & \\ -1/2 & 0 & 1/2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1/2 & 0 & 1/2 \\ & & & -1/2 & 1/2 \end{bmatrix}$$

N.B. Functional superconvergence for dual consistent formulations

A classical finite-difference SBP operator ($p = 2$)

Gives a second-order (p) approximation to the first derivative (at the near-boundary nodes) and fourth-order ($2p$) in the interior, which will produce a third-order ($p + 1$) solution to the linear convection equation (for diagonal H)

$$D = H^{-1}Q$$

$$H = \Delta x \begin{bmatrix} 17/48 & & & & & \\ & 59/48 & & & & \\ & & 43/48 & & & \\ & & & 49/48 & & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \quad \text{(Gregory rule)}$$

$$Q = \begin{bmatrix} -1/2 & 59/96 & -1/12 & -1/32 & & & & \\ -59/96 & 0 & 59/96 & 0 & & & & \\ 1/12 & -59/96 & 0 & 59/96 & -1/12 & & & \\ 1/32 & 0 & -59/96 & 0 & 8/12 & -1/12 & & \\ 0 & 0 & 1/12 & -8/12 & 0 & 8/12 & -1/12 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Boundary and Interface Conditions: Simultaneous Approximation Terms (SATs)

- weak imposition of boundary and interface conditions - penalty methods
- discontinuous solutions
- similar to discontinuous Galerkin finite-element methods

Stability

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -H^{-1}Q\boldsymbol{u} - H^{-1}(u_1 - \mathcal{U}_L)\boldsymbol{e}_1$$

where $\boldsymbol{e}_1 = [1, 0, \ldots, 0]^T$

Gives, for $\mathcal{U}_L = 0$:

$$\frac{\mathrm{d}||\boldsymbol{u}||_H^2}{\mathrm{d}t} = -2\boldsymbol{u}^T Q\boldsymbol{u} - 2\boldsymbol{u}^T(u_1)\boldsymbol{e}_1 = -(u_n^2 - u_1^2) - u_1^2 = -(u_n^2 + u_1^2) \ \leq \ 0$$

Conservation at Interfaces: Two Neighbouring Blocks

Conservation

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{u}_L}{\mathrm{d}t} &= -H_L^{-1}Q_L\boldsymbol{u}_L - H_L^{-1}(u_1^L - \mathcal{U}_L)\boldsymbol{e}_1 \\
\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t} &= -H_R^{-1}Q_R\boldsymbol{u}_R - H_R^{-1}(u_1^R - u_{n_L}^L)\boldsymbol{e}_1
\end{aligned}
$$

Gives:

$$
\begin{aligned}
\mathbf{1}^T H_L\frac{\mathrm{d}\boldsymbol{u}_L}{\mathrm{d}t} &= -(u_{n_L}^L - \mathcal{U}_L) \\
\mathbf{1}^T H_R\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t} &= -(u_{n_R}^R - u_{n_L}^L)
\end{aligned}
$$

$$
\mathbf{1}^T H_L\frac{\mathrm{d}\boldsymbol{u}_L}{\mathrm{d}t} + \mathbf{1}^T H_R\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t} = -(u_{n_R}^R - \mathcal{U}_L)
$$

Symmetric SATs are also possible

Generalized Summation by Parts (GSBP) Property

- Introduces the following three generalizations:
  - nonuniform nodal distributions
  - no repeating interior operator
  - operators without nodes on one or both boundaries

- With these generalizations it can be shown that some existing operators satisfy the GSBP property

- Enables the construction of novel operators (because no basis functions are needed explicitly)

- No longer restricted to Gregory quadrature rules

Generalized Summation by Parts Operators: Definition

An operator D is an approximation to the first derivative of degree p with the SBP property if the accuracy condition is met, H is an SPD matrix, and

$$Q + Q^T = E$$

where $(\boldsymbol{x}^i)^T E \boldsymbol{x}^j = (x_R)^{i+j} - (x_L)^{i+j},\ i,j \in [0,r],\ r \geq p$

E can be constructed as

$$E = \boldsymbol{t}_R \boldsymbol{t}_R^T - \boldsymbol{t}_L \boldsymbol{t}_L^T$$

where $\boldsymbol{t}_L$ and $\boldsymbol{t}_R$ satisfy

$$\boldsymbol{t}_L^T \boldsymbol{x}^j = x_L^j,\ \ \boldsymbol{t}_R^T \boldsymbol{x}^j = x_R^j\ , j \in [0,r]$$

This means that $\boldsymbol{t}_L^T \boldsymbol{u} \approx \mathcal{U}_L,\ \ \boldsymbol{t}_R^T \boldsymbol{u} \approx \mathcal{U}_R$ and we get

$$\begin{array}{lrcll} & \boldsymbol{v}^T E \boldsymbol{u} & \approx & \mathcal{V}_R \mathcal{U}_R - \mathcal{V}_L \mathcal{U}_L & \text{for SBP} \\ \text{and hence} & \boldsymbol{u}^T E \boldsymbol{u} & \approx & \mathcal{U}_R^2 - \mathcal{U}_L^2 & \text{for stability (with SAT)} \end{array}$$

Generalized SBP example: nonuniform nodal distribution

Hybrid Gauss-trapezoidal-Lobatto quadrature nodal distribution given for 9 nodes by ($x \in [0,1]$):

$$x = \frac{1}{8}\begin{bmatrix} 0 & \frac{12}{13} & 2 & 3 & 4 & 5 & 6 & (8-\frac{12}{13}) & 8 \end{bmatrix}^{\mathrm{T}}$$

$$D = 8\begin{bmatrix}
-\frac{864}{553} & \frac{408811}{209034} & -\frac{2901}{7742} & -\frac{278}{14931} & & & & & \\
-\frac{41}{78} & 0 & \frac{15}{26} & -\frac{2}{39} & & & & & \\
\frac{967}{7932} & -\frac{49855}{71388} & 0 & \frac{11800}{17847} & -\frac{56}{661} & & & & \\
\frac{139}{23460} & \frac{9971}{164220} & -\frac{1770}{2737} & 0 & \frac{1296}{1955} & -\frac{162}{1955} & & & \\
 & & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & & \\
 & & & \frac{162}{1955} & -\frac{1296}{1955} & 0 & \frac{1770}{2737} & -\frac{9971}{164220} & -\frac{139}{23460} \\
 & & & & \frac{56}{661} & -\frac{11800}{17847} & 0 & \frac{49855}{71388} & -\frac{967}{7932} \\
 & & & & & \frac{2}{39} & -\frac{15}{26} & 0 & \frac{41}{78} \\
 & & & & & \frac{278}{14931} & \frac{2901}{7742} & -\frac{408811}{209034} & \frac{864}{553}
\end{bmatrix}$$

Includes a standard interior operator that can be repeated

Second generalized SBP example: nonuniform nodal distribution with no repeating interior operator

Diagonal-norm operator on Chebyshev-Gauss-Lobatto quadrature nodal distribution given for 5 nodes by ($x \in [0,1]$):

$$x = \frac{1}{2}\left[\begin{array}{ccccc} -1 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 1 \end{array}\right]^{\mathrm{T}} + \left(\frac{1}{2}\right)\mathbf{1}$$

where $\mathbf{1}$ is a vector of ones

$$D = 2\left[\begin{array}{ccccc} -\frac{15}{2} & 8+2\sqrt{2} & -6 & 8-2\sqrt{2} & -\frac{5}{2} \\ -1-\frac{1}{4}\sqrt{2} & 0 & \frac{3}{2}\sqrt{2} & -\sqrt{2} & 1-\frac{1}{4}\sqrt{2} \\ \frac{1}{2} & -\sqrt{2} & 0 & \sqrt{2} & -\frac{1}{2} \\ -1+\frac{1}{4}\sqrt{2} & \sqrt{2} & -\frac{3}{2}\sqrt{2} & 0 & 1+\frac{1}{4}\sqrt{2} \\ \frac{5}{2} & -8+2\sqrt{2} & 6 & -8-2\sqrt{2} & \frac{15}{2} \end{array}\right]$$

Third generalized SBP example: nonuniform nodal distribution with no repeating interior operator and no boundary nodes

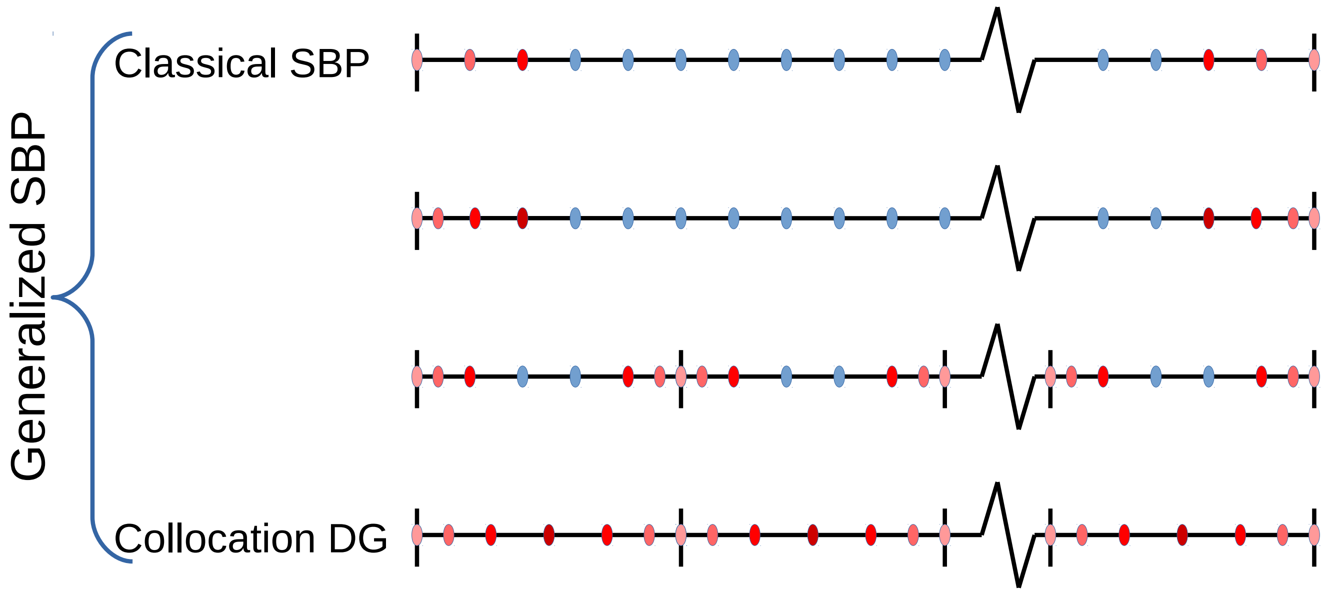
Diagonal-norm operator on Legendre-Gauss quadrature nodal distribution given for 3 nodes by ($x \in [0, 1]$):

$$\boldsymbol{x} = \frac{1}{2}\left[-\frac{\sqrt{15}}{5}, 0, \frac{\sqrt{15}}{5}\right]^{\mathrm{T}} + \left(\frac{1}{2}\right)\mathbf{1}$$

where $\mathbf{1}$ is a vector of ones

$$D = 2\begin{bmatrix} -\frac{1}{2}\sqrt{15} & \frac{2}{3}\sqrt{15} & -\frac{1}{6}\sqrt{15} \\ -\frac{1}{6}\sqrt{15} & 0 & \frac{1}{6}\sqrt{15} \\ \frac{1}{6}\sqrt{15} & -\frac{2}{3}\sqrt{15} & \frac{1}{2}\sqrt{15} \end{bmatrix}$$

Generalized SBP Operators



- classical SBP operators are applied in the traditional finite-difference manner – uniform meshes in computational space, repeating interior operator

- generalized SBP operators can be applied in an element-type manner, can have repeating interior operator or not, can have uniform or nonuniform nodal distribution

Element-type Finite-Difference Methods

	basis functions	elements	Taylor series	repeating interior operator
finite-difference method	✘	✘	✔	✔
finite-element method	✔	✔	✘	✘
element-type finite-difference method	✘	optional	✔	optional

Multidimensional GSBP Operators: some notation

In two dimensions:

$$
\begin{aligned}
S &= (x_i, y_i)_{i=1}^{n} \\
\boldsymbol{u} &= [\mathcal{U}(x_1, y_1), \ldots, \mathcal{U}(x_n, y_n)]^T
\end{aligned}
$$

Monomials of total degree p (size in 2D: $n_p^* = (p+1)(p+2)/2$):

$$
\begin{aligned}
\mathcal{P}_k(x,y) &\equiv x^i y^{j-i},\ k = j(j+1)/2 + i + 1 \ \ \forall j \in [0,1,\ldots,p],\ i \in [0,1,\ldots,j] \\
\boldsymbol{p}_k &\equiv [\mathcal{P}_k(x_1,y_1), \ldots, \mathcal{P}_k(x_n,y_n)]^T \\
\boldsymbol{p}'_k &\equiv [\frac{\partial \mathcal{P}_k}{\partial x}(x_1,y_1), \ldots, \frac{\partial \mathcal{P}_k}{\partial x}(x_n,y_n)]^T
\end{aligned}
$$

Two-Dimensional GSBP Operator: definition

1. accuracy to degree p:

$$D_x \boldsymbol{p}_k = \boldsymbol{p}'_k \ \forall k \in [1, 2, \ldots, n_p^*]$$

2. $D_x = H^{-1} Q_x$ where H is symmetric positive definite (focus on diagonal H)

3. to mimic IBP:

$$Q_x + Q_x^T = E_x, \text{which satisfies } \boldsymbol{p}_k^T E_x \boldsymbol{p}_m = \oint_\Gamma \mathcal{P}_k \mathcal{P}_m n_x \mathrm{d}\Gamma \ \ \forall k, m \in [1, 2, \ldots, n_\tau^*]$$

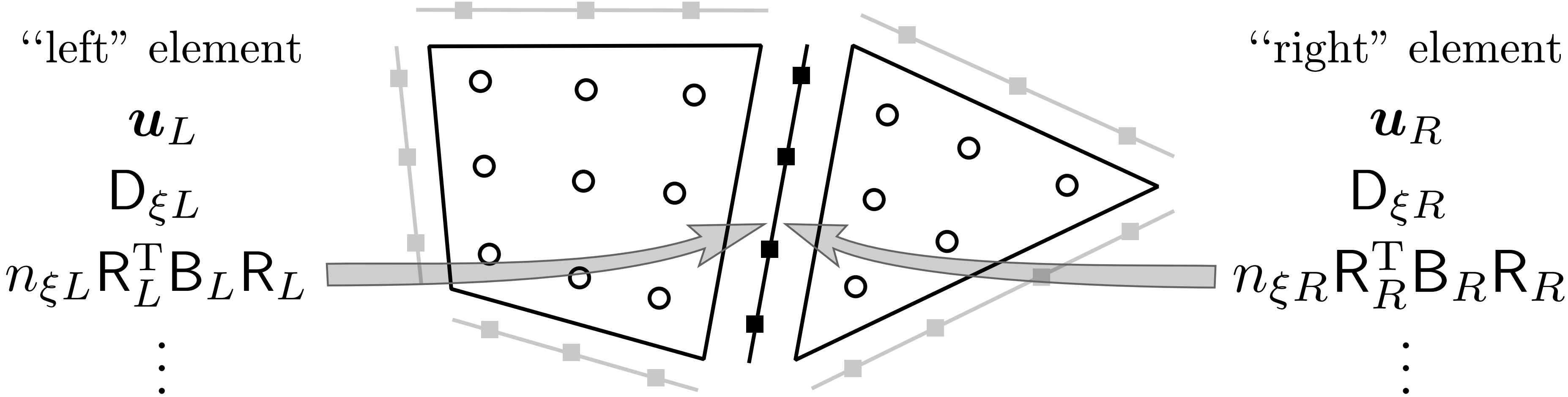
where $\tau \geq p$

Mimics integration by parts as in the 1D case:

$$\int_\Omega \mathcal{V} \frac{\partial \mathcal{U}}{\partial x} \mathrm{d}\Omega + \int_\Omega \mathcal{U} \frac{\partial \mathcal{V}}{\partial x} \mathrm{d}\Omega = \oint_\Gamma \mathcal{V}\mathcal{U} n_x \mathrm{d}\Gamma \Leftrightarrow \boldsymbol{v}^T H D_x \boldsymbol{u} + \boldsymbol{u}^T H D_x \boldsymbol{v} = \boldsymbol{v}^T E_x \boldsymbol{u}$$

(exact for $\boldsymbol{p}_k$, $\boldsymbol{p}_m$ where $k, m \leq n_p^*$)

Construction of E_ξ and Application of SATs



- open circles are the volume nodes at which the solution is stored
- black squares are the cubature nodes for the face used in decomposition of E and application of SATs
- $B_{L/R}$ hold the cubature weights
- $R_{L/R}$ are interpolation/extrapolation operators from the volume nodes to the face cubature nodes
- SATs are applied in a pointwise manner at the face cubature nodes

Construction of Operators

1. Choose or construct a symmetric cubature rule of total degree $2p-1$ on a nodal distribution with at least n_p^* nodes – gives diagonal H

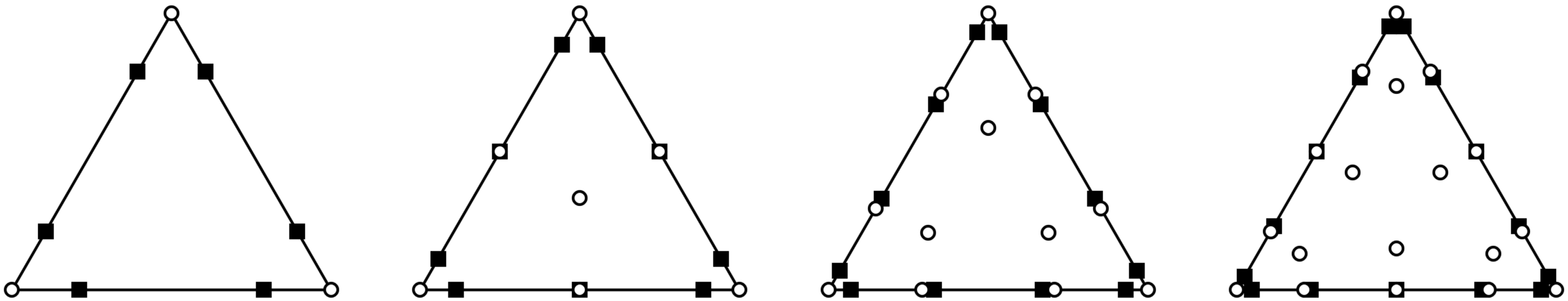
2. Choose or construct a cubature rule and nodal distribution for the faces (e.g. Legendre-Gauss with $p+1$ nodes) – gives B

3. Construct the interpolation/extrapolation operators R from volume nodes to face nodes

4. Construct matrices E_ξ and E_η

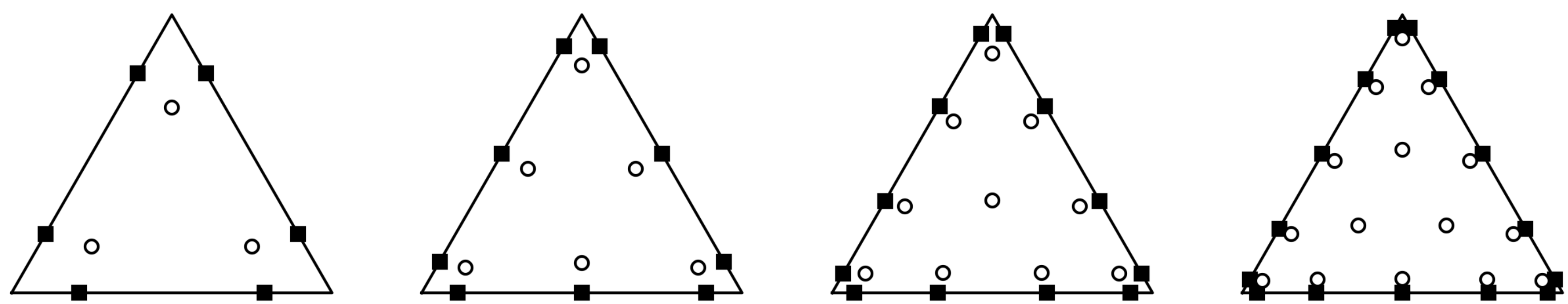
5. Determine matrices Q_ξ and Q_η from the accuracy conditions

Two Example Families of Operators



$p = 1$ $\quad$ $p = 2$ $\quad$ $p = 3$ $\quad$ $p = 4$

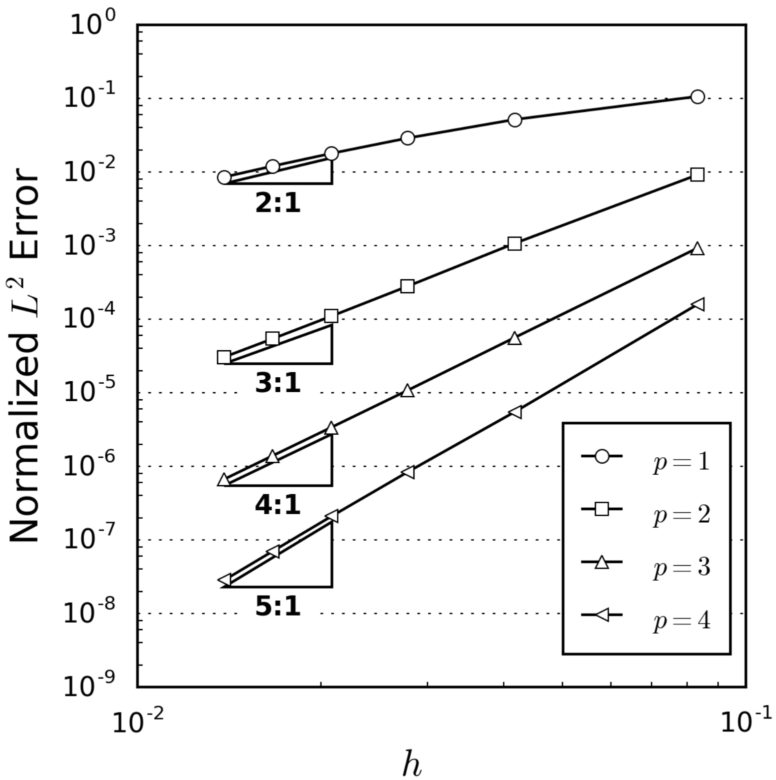
SBP-Γ family of operators with $p + 1$ nodes on each face



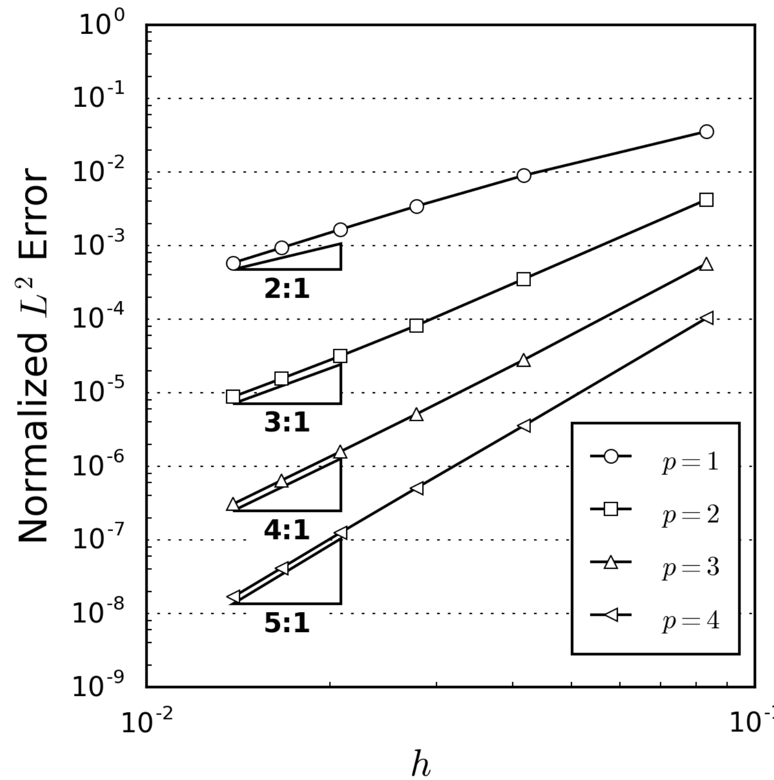
$p = 1$ $\quad$ $p = 2$ $\quad$ $p = 3$ $\quad$ $p = 4$

SBP-Ω family of operators with nodes only in the interior of the element

Accuracy (2D linear convection on curvilinear mesh)



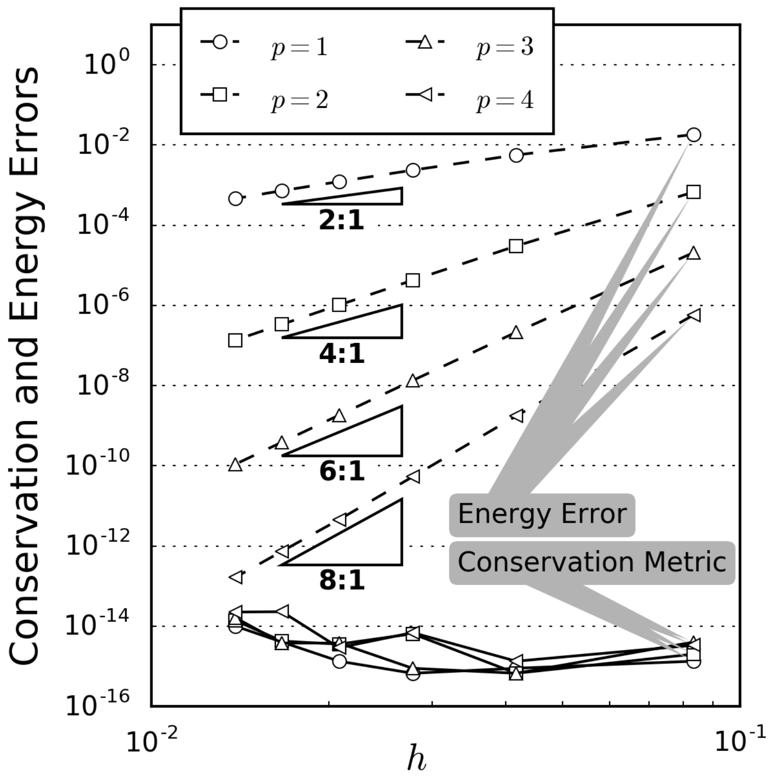
SBP-Γ family



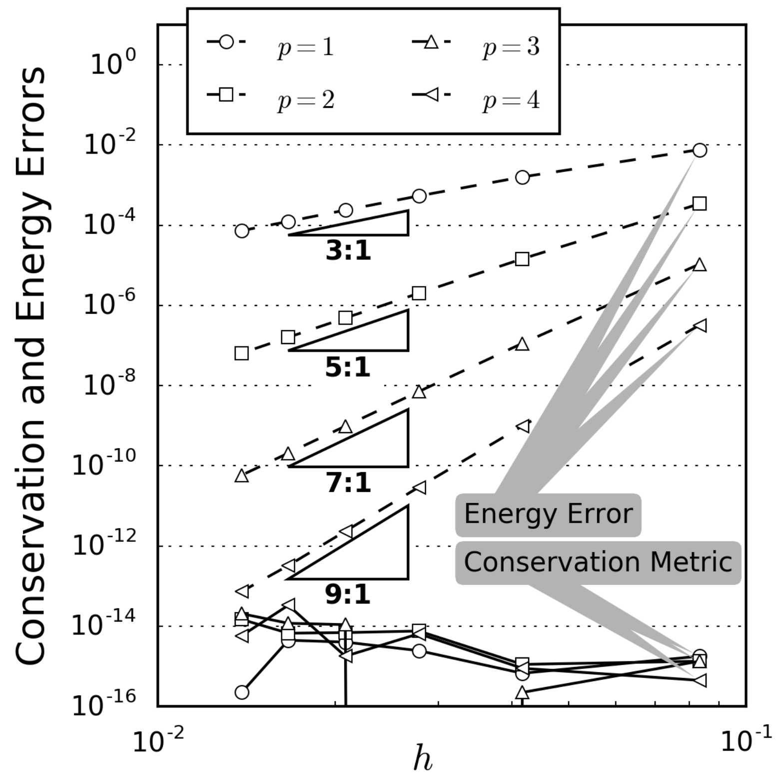
SBP-Ω family

- periodic boundary conditions, divergence-free velocity field
- convergence rates of $\mathrm{O}(h^{p+1})$ typically achieved
- consistent trends on highly nonsmooth meshes

Conservation and Stability



SBP-Γ family



SBP-Ω family

- conservative to machine precision ($\mathbf{1}^T H_g \boldsymbol{u}_0 - \mathbf{1}^T H_g \boldsymbol{u}$)
- energy stable (skew-symmetric form solved with upwind SATs)
- superlinear convergence of energy reduction ($\boldsymbol{u}_0^T H_g \boldsymbol{u}_0 - \boldsymbol{u}^T H_g \boldsymbol{u}$)

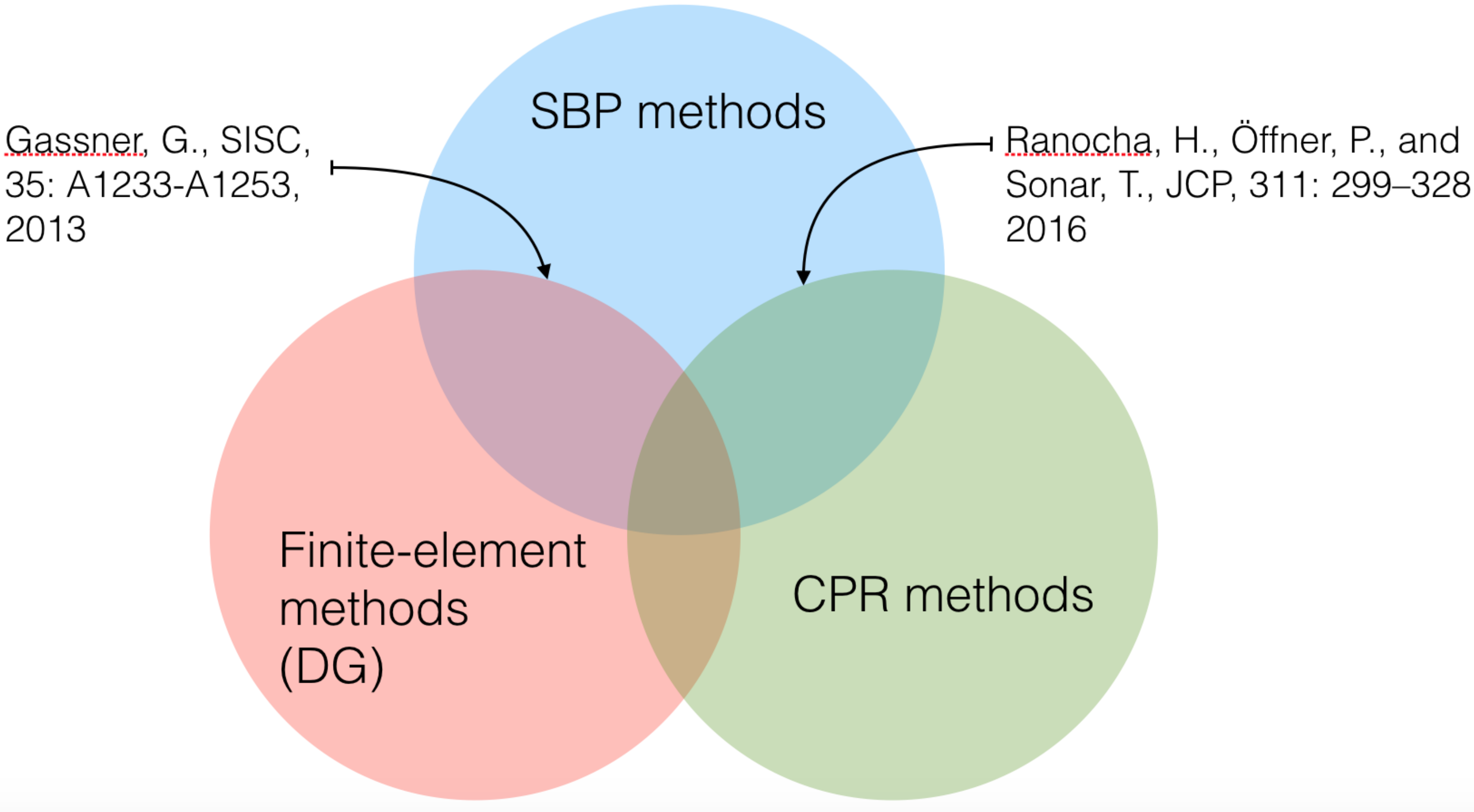
Conclusions

- The multi-dimensional summation-by-parts property is a powerful means of developing stable (and hence robust) high-order operators for unstructured grids
  - concept can be exploited in discontinuous Galerkin and flux reconstruction methods as well as in SBP operators

- Multi-dimensional summation-by-parts operators represent a potentially powerful option for high-order methods
  - very general as a result of the absence of explicit basis functions in their derivation
  - increased flexibility and generality can be exploited in various ways

- Recent extension to multi-dimensions opens up new opportunities for unstructured grids
  - much work remains to be done to unlock the potential of SBP operators!

Relationship Between SBP, DG, and CPR?



- are there methods that are advantageous in certain respects or contexts that lie only within the SBP circle?

Further Reading

Crean, J., Hicken, J.E., Del Rey Fernandez, D.C., Zingg, D.W., and Carpenter, M.H., "High-Order, Entropy Stable Discretizations of the Euler Equations for Complex Geometries," AIAA Paper 2017-4496, Denver, June 2017, submitted to J. Comp. Phys. (on arXiv).

Friedrich, L., Del Rey Fernandez, D.C., Winters, A.R., Gassner, G.J., Zingg, D.W., and Hicken, J.E., "Conservative and Stable Degree Preserving SBP Finite Difference Operators for Non-Conforming Meshes," submitted to J. of Scientific Computing, November 2016 (on arXiv).

Del Rey Fernandez, D.C., Hicken, J.E., and Zingg, D.W., "Simultaneous Approximation Terms for Multi-Dimensional Summation-By-Parts Operators," accepted by Journal of Scientific Computing, July 2017 (on arXiv).

Hicken, J.E., Del Rey Fernandez, D.C., and Zingg, D.W., "Multidimensional Summation- By-Parts Operators: General Theory and Application to Simplex Elements," SIAM Journal on Scientific Computing, 38-4, 2016, pp. A1935-A1958.

Del Rey Fernandez, D.C., and Zingg, D.W., "Generalized Summation-By-Parts Operators for the Second Derivative with a Variable Coefficient," SIAM J. on Scientific Computing, Vol. 37(6), 2015, pp. A2840-A2864.

Del Rey Fernandez, D.C., Boom P.D., and Zingg, D.W., "A Generalized Framework for Nodal First Derivative Summation-By-Parts Operators," J. of Computational Physics, Vol. 266, June 2014, pp. 214-239.